RESOLUTION OF SPECTRAL LINES OF UNEQUAL INTENSITY

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ABSTRACT. In this paper the authors have discussed the dependence of resolving power on intensity ratio of the two lines to be resolved, detecting instrument and the stage of resolution desired when (i) instrumental line width is negligible and (ii) when natural line width is negligible.

INTRODUCTION

In general the spectral lines, sought to be resolved are not of equal intensity. This fact has not been given sufficient importance and there are no results, regarding the variation of resolving powers of instruments with the intensity ratio of the two spectral lines to be resolved, except those of Sparrow (1916) and Sodha (1952) for non-absorbing prism. The latter's results are not correct since he took the central minimum as the point of intersection of the component two intensity patterns.

Sparrow (1916) has suggested that two spectral lines, to be resolved, should have no dip in the resultant intensity pattern at the limit of resolution. In other words. resolution to non-resolution occurs, when the central minimum of the resultant intensity pattern of the two lines, just vanishes. It is obvious that the visibility

$$V = \frac{I_{max}-I_{min}}{I_{max}+I_{min}}$$

becomes zero at the limit of resolution, when Sparrow's criterion is applied. Hence Sparrow's criterion merely sets an upper limit for the resolving power.

Tolansky (1947) modified the Rayleigh criterion, when the spectral lines are of unequal intensity and stated that the two lines can be resolved when they are separated such that at the point of overlap the intensity of the stronger has fallen to two-fifths of the maximum intensity of the weaker line. In this modification Tolansky has oviously overlooked the following facts.

(i) The weaker maximum of the resultant intensity pattern is different from the maximum intensity of the weaker line. The tail of the stronger line also contributes to it.

(ii) For lines of unequal intensity, the central minimum does not occur at the point of intersection of the two component intensity patterns.

Keeping the above facts in view, Sodha (1952) modified the Rayleigh criterion for resolution of lines of unequal intensity and stated that at the limit of resolution the central minimum of the resultant intensity pattern should be $8/\pi^2$ times the weaker maximum.

Ditchburn (1930) has pointed out that the value of $(I_{min}/I_{max}) = C$, of the resultant intensity pattern at limiting resolution is characteristic of the detecting instrument and the stage of resolution desired. It may be added that in discussing resolution of lines of unequal intensity I_{max} should refer to the weaker maximum of the resultant intensity pattern.

In this communication, the authors have investigated the variation of resolving power with the intensity ratio $b\ (> 1)$ of the two spectral lines to be resolved for various values of C $(0.4 \leqslant C \leqslant 0.98)$ in the following two cases :

(i) When instrumental width is negligible and the intensity distribution of the line is governed by Doppler effect.

(ii) When Doppler width is negligible and the intensity distribution is governed by the instrument (Fabry Perot etalon).

NEGLIGIBLE INSTRUMENTAL WIDTH

The intensity distribution of a spectral line of wave number ν_0 due to Doppler effect is given by

$$I' = I_0\, e^{-\beta(\nu-\nu_0)^2}$$

where $\beta = \mu c^2{}_0/2RT\nu_0{}^2$, μ being the mass of radiant atoms.

The intensity distribution of another spectral line of wave number $\nu_0+\Delta\nu$ and an intensity b times the first is given by

$$I'' = bI_0\, e^{-\beta(\nu-\nu_0-\Delta\nu)^2}$$

if $\Delta\nu$ is small (β same for both lines)

Putting $\sqrt{\beta}(\nu-\nu_0) = x$ and $\sqrt{\beta}.\Delta\nu = a$, the resultant intensity pattern is given by

$$\frac{I}{I_0} = e^{-x^2} + be^{-(x-a)^2} \qquad \ldots \quad (1)$$

Neglecting shrinkage effect, the weaker maximum of the resultant intensity pattern ($x_{max} = 0$) is given by—

$$\frac{I_{max}}{I_0} = 1+be^{-a^2} \qquad \ldots \quad (2)$$

The value of $x (= x_{min})$ for which the minimum of the resultant intensity pattern occurs, is given by

$$\frac{1}{I_0}\frac{dI}{dx} = -xe^{-x^2} + b(a-x)e^{-(a-x)^2} = 0$$

or

$$\phi(x) = b\phi(a-x) \qquad \ldots \quad (3)$$

where $\phi(x) = xe^{-x^2}$

For $b = 1$, Eqn. (3) gives

$$x_{min} = a/2 \qquad (3a)$$

The minimum of the resultant intensity pattern is given by

$$\frac{I_{min}}{I_0} = e^{-x^2_{min}} + be^{-(a-x_{min})^2} \qquad \dots (4)$$

For optimum resolution

$$C = \frac{I_{min}}{I_{ma}} \qquad \dots (5)$$

The resolving power is given by

$$\frac{\lambda}{d\lambda} = \frac{\nu_0}{\Delta\nu} = \frac{\sqrt{\beta}}{a} \cdot \nu_0 = \alpha . C_0\sqrt{\frac{\mu}{2RT}} \qquad \dots (6)$$

where

$$\alpha =$$

The details of calculation are given in Table I which gives the variation of α with C for $b = 1, 2, 3, 4$ and 5. The table is illustrated by figure 1.

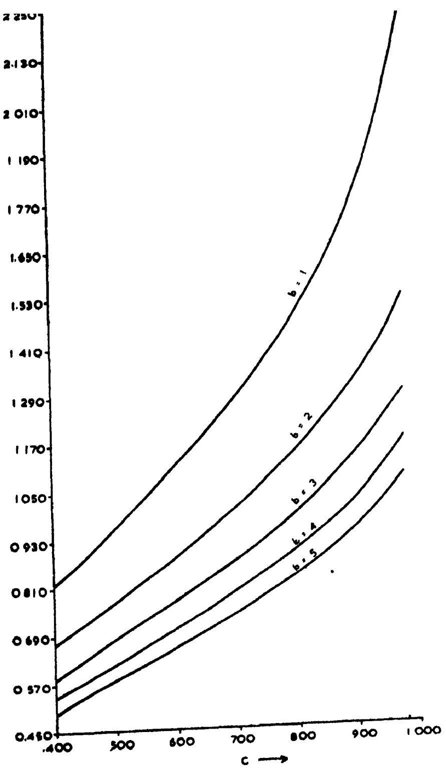

Fig. 1. Variation of α with C when instrumental width is negligible.

TABLE I

Variation of α with C for $b = 1, 2, 3, 4$ and 5 when instrumental width is negligible.

b	x_{min}	$\phi(x_{min})$	$\phi(a=x_{min})$ $\frac{1}{b}\phi(x_{min})$	$a - x_{min}$	$a = x_{min} + (a - x_{min})$	$\frac{I_{min}}{I_o}$	$\frac{I_{max}}{I_o}$	$C = \frac{I_{min}}{I_{max}}$	α
1	0.806	—	—	0.806	1.612	1.0427	1.0743	0.971	0.620
1	0.837	—	—	0.837	1.674	0.9932	1.0608	0.937	0.597
1	0.894	—	—	0.894	1.788	0.8986	1.0400	0.863	0.559
1	1.000	—	—	1.000	2.000	0.7358	1.0183	0.722	0.500
1	1.049	—	—	1.049	2.098	0.6658	1.0123	0.658	0.477
1	1.095	—	—	1.095	2.190	0.6024	1.0083	0.597	0.457
1	1.183	—	—	1.183	2.366	0.4932	1.00370	0.491	0.423
1	1.225	—	—	1.225	2.450	0.4462	1.00248	0.445	0.408
2	0.632	0.4239	0.2119	1.365	1.997	0.9817	1.0368	0.947	0.501
2	0.775	0.4252	0.2126	1.363	2.138	0.8602	1.0206	0.843	0.468
2	0.894	0.4018	0.2009	1.391	2.285	0.7395	1.0108	0.732	0.439
2	1.000	0.3679	0.1839	1.433	2.433	0.6250	1.0054	0.622	0.411
2	1.140	0.3106	0.1553	1.457	2.597	0.5125	1.0024	0.511	0.385
2	1.225	0.2733	0.1366	1.561	2.786	0.3974	1.0009	0.397	0.359
3	0.548	0.4057	0.1352	1.565	2.113	0.9996	1.0347	0.966	0.473
3	0.707	0.4289	0.1430	1.542	2.249	0.8841	1.0190	0.868	0.445
3	0.837	0.4155	0.1385	1.555	2.392	0.7633	1.0099	0.756	0.418
3	1.049	0.3492	0.1164	1.623	2.672	0.5491	1.0023	0.548	0.374
3	1.140	0.3106	0.1035	1.667	2.807	0.4586	1.0011	0.458	0.356
3	1.183	0.2917	0.0972	1.690	2.873	0.4184	1.0008	0.418	0.348
4	0.447	0.3661	0.0915	1.712	2.159	1.0323	1.0377	0.995	0.465
4	0.548	0.4057	0.1014	1.674	2.222	0.9840	1.0286	0.957	0.450
4	0.632	0.4239	0.1060	1.658	2.290	0.9257	1.0212	0.906	0.437
4	0.775	0.4252	0.1063	1.657	2.432	0.8046	1.0109	0.796	0.411
4	0.894	0.4018	0.1005	1.678	2.572	0.6877	1.0005	0.687	0.389
4	1.000	0.3679	0.0920	1.710	2.710	0.5836	1.0000	0.584	0.369
4	1.095	0.3298	0.0825	1.748	2.843	0.4888	1.0012	0.488	0.352
4	1.140	0.3106	0.0777	1.768	2.908	0.4471	1.0008	0.447	0.344
4	1.183	0.2917	0.0729	1.789	2.972	0.4098	1.0005	0.409	0.336
5	0.447	0.3661	0.0732	1.788	2.235	1.0227	1.0337	0.989	0.447
5	0.632	0.4239	0.0848	1.738	2.370	0.9144	1.0181	0.898	0.422
5	0.775	0.4252	0.0850	1.737	2.512	0.7929	1.0091	0.786	0.398
5	0.894	0.4018	0.0804	1.756	2.650	0.6792	1.0054	0.676	0.377
5	1.000	0.3679	0.0736	1.786	2.786	0.5727	1.0021	0.571	0.359
5	1.095	0.3298	0.0660	1.823	2.918	0.4820	1.0010	0.482	0.343
5	1.183	0.2917	0.0583	1.861	3.044	0.4038	1.0005	0.403	0.329

*Obtained by trial and error.

The variation of α with b for various values of C is given in Table II, which has been tabulated from figure 1.

TABLE II

Variation of α with b for various values of C when instrumental width is negligible.

C ↓ | b →	1	2	3	4	5
0.4	0.393	0.359	0.345	0.335	0.328
0.5	0.425	0.383	0.364	0.354	0.346
0.6	0.458	0.406	0.384	0.372	0.363
0.7	0.492	0.430	0.405	0.391	0.381
0.8 Rayleigh's criterion	0.529	0.456	0.428	0 412	0.401
0.9	0.576	0.484	0.454	0.435	0.423
0.98 Abbe's criterion	0.624	0.515	0.478	0.457	0.444

Table II has been illustrated by figure 2.

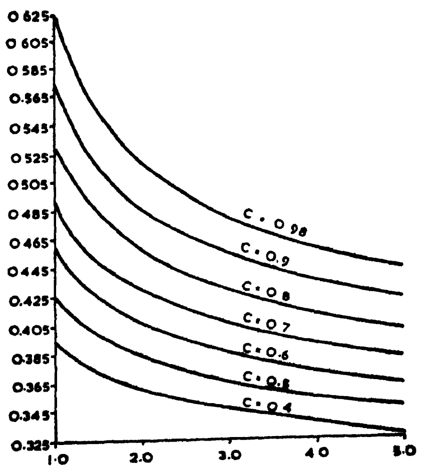

Fig. 2. Variation of α with b, when instrumental width is negligible.

FABRY PEROT ETALON

The intensity pattern of a spectral line in the order (n_0+n), where n is a fraction and n_0 an integer is given for Fabry Perot etalon by

$$I' = \frac{I_0}{1+F\sin^2\pi(n_0+n)} = \frac{I_0}{1+x^2}$$

where F is the coefficient of fineness and $x = \pi n F^{\frac{1}{2}}$.

The intensity distribution of another spectral line, separated by an order Δn, of an intensity b times the first is given by

$$I'' = \frac{bI_0}{1+F\sin^2\pi(n_0+n-\Delta n)} = \frac{bI_0}{1+(x-a)^2}$$

where $$a = \pi\Delta n F^{\frac{1}{2}}$$

The resultant intensity pattern is given by

$$\frac{I}{I_0} = \frac{1}{1+x^2}+\frac{b}{1+(x-a)^2} \quad \ldots (7)$$

The values of x for which the maxima or minimum of the resultant intensity pattern occur are given by

$$-\frac{1}{2I_0}\frac{\partial I}{\partial x} = \frac{x}{(1+x^2)^2}-\frac{b(a-x)}{\{1+(a-x)^2\}^2} = 0$$

or $$F(x) = bF(a-x) \quad \ldots (8)$$

where $F(x) = \dfrac{x}{(1+x^2)^2}$

The weaker maximum will occur near $x = 0$ and can be obtained by solving Eqn. (8) by the method of successive approximations given by Sodha (1955).

The minimum and weaker maximum of the resultant intensity pattern are given by

$$\frac{I_{min}}{I_0} = \frac{1}{1+x^2_{min}}+\frac{b}{1+(a-x_{min})^2} \quad \ldots (9)$$

and

$$\frac{I_{max}}{I_0} = \frac{1}{1+x^2_{max}}+\frac{b}{1+(a-x_{max})^2} \quad \ldots (10)$$

For optimum resolution

$$C = \frac{I_{min}}{I_{max}}$$

The resolving power is given by

$$\frac{\lambda}{d\lambda} = \frac{n_0}{\Delta n} = \frac{\pi}{a} \cdot n_0 F^{\frac{1}{2}} = \alpha \, n_0 F^{\frac{1}{2}} \qquad \dots \quad (11)$$

where

$$\alpha = \pi / a$$

The details of calculation are given in Table III, which gives the variation of α with C for $b = 1, 2, 3, 4$ and 5. The table is illustrated by figure 3.

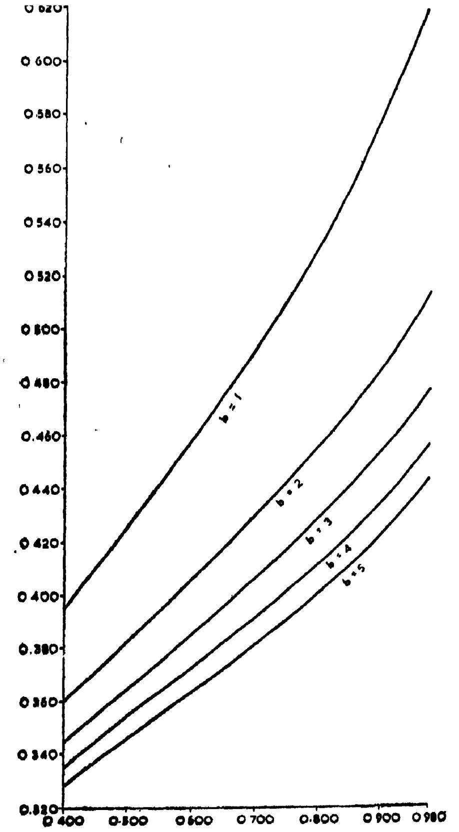

Fig. 3. Variation of α with C for F. P. etalon.

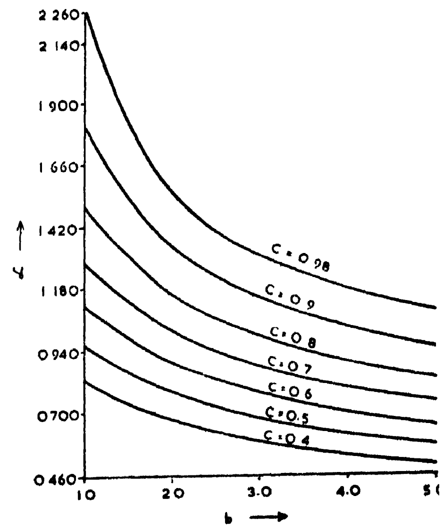

Fig. 4. Variation of α with b for F. P. etalon.

TABLE III

Variation of α with C for $b = 1, 2, 3, 4$ and 5 for F.P. etalon.

b	x_{min}	$F(x_{min})$	$F(a-xI_{min}) = \frac{1}{b}F(x_{min})$	$a-x_{min}$	$a = x_{min} + a - x_{min}$	x_{max}	$\frac{I_{min}}{I_0}$	$\frac{I_{max}}{I_0}$	$C = \frac{I_{min}}{I_{max}}$	α
1	0.70	—	—	0.70	1.4	0.229	1.3423	1.3719	0.978	2.24*
1	0.80	—	—	0.80	1.6	0.155	1.2195	1.3004	0.934	1.964
1	0.90	—	—	0.90	1.8	0.117	1.1050	1.2474	0.886	1.746
1	1.00	—	—	1.00	2.0	0.090	1.0000	1.2071	0.828	1.571
1	1.15	—	—	1.15	2.3	0.062	0.8611	1.1626	0.741	1.366
1	1.25	—	—	1.25	2.5	0.050	0.7805	1.1403	0.684	1.257
1	1.40	—	—	1.40	2.8	0.037	0.6757	1.1144	0.606	1.122
1	1.60	—	—	1.60	3.2	0.0255	0.5618	1.0897	0.516	0.982
1	1.85	—	—	1.85	3.7	0.0174	0.4522	1.0684	0.423	0.849
1	1.90	—	—	1.90	3.8	0.016	0.4338	1.0650	0.407	0.827
2	0.65	0.321	0.1605	1.395	2.045	0.210	1.382	1.416	0.976	1.536
2	0.80	0.297	0.1485	1.460	2.260	0.145	1.247	1.344	0.928	1.390
2	1.00	0.250	0.1250	1.610	2.610	0.096	1.0564	1.271	0.831	1.204
2	1.20	0.201	0.1005	1.800	3.000	0.065	0.8816	1.2038	0.732	1.047
2	1.40	0.159	0.0795	2.010	3.410	0.044	0.7346	1.1602	0.633	0.921
2	1.60	0.126	0.0630	2.220	3.820	0.032	0.6183	1.1293	0.548	0.823
2	1.80	0.100	0.0500	2.440	4.240	0.024	0.5234	1.1059	0.473	0.741
2	2.00	0.080	0.0400	2.680	4.680	0.020	0.4445	1.0876	0.409	0.671
3	0.70	0.315	0.1050	1.750	2.460	0.200	1.4033	1.4527	0.966	1.277
3	0.80	0.297	0.0990	1.810	2.610	0.155	1.3114	1.4035	0.934	1.204
3	1.00	0.250	0.0833	1.960	2.960	0.100	1.1196	1.3169	0.850	1.061
3	1.20	0.201	0.0670	2.170	3.370	0.070	0.9354	1.2474	0.750	0.932
3	1.40	0.159	0.0530	2.390	3.790	0.050	0.7848	1.1977	0.655	0.829
3	1.60	0.126	0.0420	2.630	4.230	0.0365	0.6598	1.1601	0.569	0.743
3	1.80	0.100	0.0333	2.900	4.700	0.027	0.5546	1.1307	0.490	0.669
3	2.00	0.080	0.0266	3.140	5.140	0.0205	0.4753	1.1099	0.428	0.611
4	0.67	0.319	0.0797	2.000	2.670	0.220	1.4902	1.5253	0.977	1.177
4	0.70	0.315	0.0787	2.015	2.715	0.215	1.4616	1.5078	0.969	1.157
4	0.90	0.275	0.0687	2.140	3.040	0.140	1.2694	1.4059	0.903	1.033
4	1.20	0.201	0.0503	2.450	3.650	0.075	0.9813	1.2846	0.764	0.861
4	1.50	0.142	0.0355	2.810	4.310	0.046	0.7573	1.2066	0.628	0.729
4	1.80	0.100	0.0250	3.200	5.000	0.030	0.5917	1.1610	0.510	0.628
4	2.00	0.080	0.0200	3.500	5.500	0.023	0.5019	1.1285	0.445	0.571
4	2.20	0.064	0.0160	3.800	6.000	0.017	0.4303	1.1084	0.388	0.524
5	0.77	0.303	0.0607	2.260	3.030	0.185	1.4465	1.5167	0.954	1.037
5	0.90	0.275	0.0550	2.350	3.250	0.142	1.3191	1.4495	0.910	0.967
5	1.20	0.201	0.0402	2.670	3.870	0.081	1.0249	1.3191	0.777	0.812
5	1.50	0.142	0.0284	3.070	4.570	0.049	0.7873	1.2307	0.640	0.688
5	1.80	0.100	0.0200	3.500	5.300	0.032	0.6132	1.1729	0.523	0.593
5	2.10	0.072	0.0144	3.940	6.040	0.0215	0.4874	1.1338	0.430	0.520
5	2.30	0.058	0.0116	4.260	6.560	0.0170	0.4201	1.1138	0.377	0.479

*obtained by trial and error.

The variation of α with b for various values of C is given in Table IV, which has been tabulated from figure 3.

TABLE IV

Variation of α with b for various values of C, for F.P. etalon

C / $b \rightarrow$	1	2	3	4	5
0.4	0.822	0.666	0.585	0.534	0.492
0.5	0.954	0.768	0.678	0.618	0.576
0.6	1.110	0.882	0.774	0.702	0.654
0.7	1.284	1.005	0.876	0.798	0.738
0.8 Rayleigh's criterion	1.500	1.152	0.990	0.906	0.834
0.9	1.800	1.338	1.146	1.026	0.954
0.98 Abbe's criterion	2.250	1.548	1.308	1.194	1.092

Table IV is illustrated by figure 4.

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REFERENCES

Sparrow, 1916, *Astrophys. J.*, **44**, 76.
Ditchburn, 1930, *Proc. Roy. Irish. Acad.*, **39**, 58.
Tolansky, 1947, High Resolution Spectroscopy (Methuen & Co., London), 87.
Sodha, 1952, *Sc. & Cult.*, **18**, 248.
Sodha, 1955, *Ind. J. Phys.*, **29**, 461.